§5.7 Applications to Differential Equations

Recall from calculus if $y(t)$ is a differentiable real-valued function with

$$y' = k \cdot y$$

for some constant k, then

$$y(t) = C \cdot e^{kt}$$

where $C = y(0)$. In general a <u>differential equation</u> is an equation relating y and its derivative y' (or higher order derivatives $y^{(n)}$)

More on this in MA 266.

We consider <u>systems of differential equations</u>.

Let $X_1, X_2, \ldots, X_n$ be differentiable functions of variable t. Write $X_1', X_2', \ldots, X_n'$ for their derivatives.

Consider the system

$$\left\{\begin{array}{l} X_1' = a_{11} X_1 + a_{12} X_2 + \cdots + a_{1n} X_n \\ X_2' = a_{21} X_1 + a_{22} X_2 + \cdots + a_{2n} X_n \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ X_n' = a_{n1} X_1 + a_{n2} X_2 + \cdots + a_{nn} X_n \end{array}\right.$$

For some constants a_{ij}. Notice we can write this as a matrix equation

$$X'(t) = A \cdot x(t)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad X'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

A <u>solution</u> of this equation is a vector-valued function satisfying $X' = A \cdot X$

In general we are interested in the set of solutions to these equations.

Notice

1) $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is always a solution to $X' = A \cdot X$

2) If u, v are solutions ($v' = Av$ and $u' = Au$) then $u+v$ is as well:

$$A(u+v) = Au + Av = u' + v' = (u+v)'$$

3) If v is a solution and c a scalar, then $c \cdot v$ is a solution as well

$$A \cdot cv = c \cdot Av = c \cdot v' = (c \cdot v)'$$

With this, the set of all solutions of $x' = A \cdot x$ forms a subspace of the vector space of all $\mathbb{R}^n$-valued functions.

We call a basis for this vector space of solutions a <u>fundamental set of solutions</u>. If A is $n \times n$, there are n fundamental solutions, hence this space is n-dimensional.

In general, we wish to compute the fundamental set of solutions to $x' = Ax$. However we can also determine a unique solution given an <u>initial value</u> $x(0) = x_0$, for some vector x_0 of $\mathbb{R}^n$.

<u>Example</u>

Suppose $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and we wish to find a fundamental set of solutions.

Notice this is equivalent to

$$\begin{cases} x_1' = 2x_1 \\ x_2' = -3x_2 \end{cases} \implies \begin{cases} x_1 = c_1 e^{2t} \\ x_2 = c_2 e^{-3t} \end{cases}$$

for some constants c_1, c_2

Thus

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$$

so

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t} \right\}$$

is a fundamental set of solutions.

In this last example we say the system is decoupled since each derivative depended only on the corresponding function. Equivalently because A was diagonal.

Diagonal matrices are easy to work with in general so eventually we'll have to diagonalize matrices to produce decoupled systems.

Remark

In general solutions to $x' = Ax$ are linear combinations of functions of the form

$$x(t) = v \cdot e^{\lambda t}$$

Notice that if $x(t) = v e^{\lambda t}$ is a solution, then

$$x'(t) = \lambda v e^{\lambda t}$$ (chain rule and v a constant vector)

and

$$A\,x(t) = A v e^{\lambda t}$$

Thus $x'(t) = A\,x(t)$ if and only if $Av = \lambda v$.
In other words λ is an eigenvalue of A with associated eigenvector v.

These fundamental solutions are sometimes called the <u>eigenfunctions</u> of $x' = Ax$.

<u>Example</u>

suppose $x'(t) = A \cdot x(t)$ where $A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$

a) Find a fundamental set of solutions

b) Solve the initial value problem

$$\begin{cases} x' = A \cdot x & \text{(with } A \text{ as above)} \\ x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$$

solution:

a) $\det(A-\lambda I) = \det\begin{bmatrix} 7-\lambda & -1 \\ 3 & 3-\lambda \end{bmatrix} = (7-\lambda)(3-\lambda)+3$

$$= \lambda^2 - 10\lambda + 24$$

$$= (\lambda-4)(\lambda-6)$$

Thus $\lambda_1 = 4$ and $\lambda_2 = 6$ are the eigenvalues

• $\lambda_1 = 4$: $A-4I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$

computing a basis for Null$(A-4I)$ we find
$V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a corresponding eigenvector

• $\lambda_2 = 6$: $A-6I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

so $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Thus

$$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} \right\}$$

is a fundamental set of solutions.

Cont.

b) We know all solutions look like

$$x(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

Just need to find c_1 and c_2 if $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \underbrace{e^0}_{=1} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{e^0}_{=1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 5/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \end{array}\right]$$

$$x(t) = -\frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is the unique solution to the initial value problem.